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Kummer Elements and the mod-3 Invariant of Albert Algebras

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Let k be a field with characteristic not 2 and 3. Assume that k contains the cube roots of unity. Let J be a Tits'-first-construction Albert division algebra over k. In this paper we relate Kummer elements in J with the mod-3 invariant $g_3(J)$. We prove that if $x \in J$ is a Kummer element with $x^3 = \lambda$, then $J \simeq J(D, \lambda)$ for some D, a degree-3 central division algebra over k. We show that if $J_1 = J(A, \mu)$ and $J_2 = J(B, \nu)$ are Tits' first-construction Albert division algebras with $g_3(J_1) = g_3(J_2)$ then $J_2 \simeq J(D, \mu)$ for some degree-3 central division algebra D over k.

INTRODUCTION

One understands the mod-2 invariants of Albert algebras fairly well; for example, the invariant f_3 divides the invariant f_5 (cf. [KMRT]). We want to understand what symbols occur in the decomposition of the mod-3 invariant g_3 of a given Albert algebra J over k. It suffices to do this for Albert division algebras arising from Tits' first construction, since g_3 is a decomposable element in $H^3(k, \mathbb{Z}/3)$ (cf. [T, Remark 3; KMRT, (40.9)]). Let k be a field with characteristic different from 2 and 3 and assume that k contains all the cube roots of unity. We prove (Corollary 3) the following equality of sets:

$$\{\mu \in k^* \mid J \simeq J(A, \mu)\} = \{\mu \in k^* \mid x^3 = \mu, x \in J\}.$$

In other words, the symbols occurring in the decomposition of g_3 correspond to the norms of "Kummer elements" in J.

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Let A be a central division algebra over k of degree 3. Then, by a theorem of Wedderburn, A is cyclic [W]. Let A^+ denote the special Jordan algebra corresponding to A. Then by [PR1, (2.7)], it is a first Tits' construction. Following Rost [R2], we call an element $x \in A^+$ a *Kummer element* if the characteristic polynomial of x is of the form $P(X) = X^3 - \lambda$ for some $\lambda \in k^*$. By the Skolem–Noether theorem it follows that given a Kummer element $x \in A^+$ with $x^3 = \lambda$, one can find a cubic cyclic extension L/k in A^+ such that A^+ is the first Tits' construction $J(L, \lambda)$. Let J be an Albert division algebra over k, arising from Tits' first construction. One can ask, in the spirit of the above discussion, whether for a Kummer element $x \in J$ (defined analogously) with $x^3 = \lambda$ there exists a central division algebra D of degree 3 over k such that $J \simeq J(D, \lambda)$. In this note, we prove that indeed this is the case (Theorem). We prove that for Albert division algebras $J_1 = J(A, \mu)$ and $J_2 = J(B, \nu)$, if $g_3(J_1) = g_3(J_2)$ then $J_2 \simeq J(C, \mu)$ for some central division algebra C of degree 3 over k (Corollary 2), where $g_3(J_i)$ is the mod-3 invariant of J_i.

PRELIMINARIES

We briefly describe the first Tits' construction of Albert algebras. Let k be a field with characteristic different from 2 and 3. Let A be a central simple algebra over k of degree 3 and let $\mu \in k^*$ be arbitrary. Let

$$J(A, \mu) = A_0 \oplus A_1 \oplus A_2, \quad A_i = A,$$

for $i = 0, 1, 2$ and define a multiplication on $J(A, \mu)$ by

$$\begin{aligned}(a_0, a_1, a_2)(a_0', a_1', a_2') = (&a_0.a_0' + \overline{a_1 a_2'} + \overline{a_1' a_2}, \overline{a_0} a_1' + \overline{a_0'} a_1 \\ &+ \mu^{-1} a_2 \times a_2', a_2' \overline{a_0} + a_2 \overline{a_0'} + \mu a_1 \times a_1'),\end{aligned}$$

where

$$a.b = \frac{1}{2}(ab + ba), \quad a \times b = a.b - \frac{1}{2}(t(a)b + t(b)a) + \frac{1}{2}(t(a)t(b) - t(a.b)),$$

$$\overline{a} = \frac{1}{2}(t(a) - a),$$

t denoting the reduced trace on A. With this multiplication, $J(A, \mu)$ is an Albert algebra over k. It is known to be a division algebra if and only if A is a division algebra and μ is not a norm from A ([J, Chap. IX, Theorem 20]). One can do the same construction as above, replacing A with a cubic field extension E of k. Then one arrives at a special Jordan algebra $J(E, \mu)$ of degree 3, which is of the form A^+ for a degree-3 central simple algebra, provided E is cyclic ([PR1, (2.7)]).

We need to define the mod-3 invariant (also known as the Serre–Rost invariant) for Albert algebras. Let J be an Albert algebra arising from Tits' first construction and let $J = J(A, \mu)$, as described above. Then the mod-3 invariant $g_3(J) \in H^3(k, \mathbb{Z}/3)$ of J is defined by

$$g_3(J) = [A] \cup [\mu],$$

where $[A] \in H^2(k, \mathbb{Z}/3)$ is the Brauer class of A and $[\mu] \in H^1(k, \mathbb{Z}/3)$ is the class of μ. Rost has proved that $g_3(J)$ is well defined and is compatible with base change. Further, J is a division algebra if and only if $g_3(J) \neq 0$ ([R1]).

We also quickly recall some features of the second Tits' construction of Albert algebras. Here one starts with a quadratic extension K of k and a central simple algebra B of degree 3 over K with a unitary involution σ over K/k and $u \in B^*$ with $\sigma(u) = u$ and $\mathrm{Nrd}(u) = \mu\bar{\mu}$ for $\mu \in K^*$, where Nrd denotes the reduced norm map on B. To this datum, one attaches an Albert algebra $J(B, \sigma, u, \mu) = (B, \sigma)_+ \oplus B$, where $(B, \sigma)_+ = \{x \in B \mid \sigma(x) = x\}$. This is a form for the first Tits' construction $J(B, \mu)$ over K; i.e., one has $J(B, \sigma, u, \mu) \otimes K \simeq J(B, \mu)$ as algebras over K. One knows that $J(B, \sigma, u, \mu)$ is a division algebra if and only if B is a division algebra and μ is not a reduced norm from B. One defines the mod-3 invariant for $J = J(B, \sigma, u, \mu)$ as $g_3(J) = -\mathrm{Cor}_{K/k}([B] \cup [\mu]) \in H^3(k, \mathbb{Z}/3)$. This is well defined and is compatible with base change ([R1]). One knows also that the mod-3 invariant g_3 is a decomposable element in $H^3(k, \mathbb{Z}/3)$, i.e., it is a cup product of H^1-classes (cf. [T, Remark 3; KMRT, (40.9)]).

RESULTS

Throughout this note, we work over a base field k with a characteristic different from 2 and 3 and we assume that k contains the cube roots of unity. We call $0 \neq x \in J$, with J an Albert division algebra over k, a *Kummer element* if $x^3 = \lambda$ for some $\lambda \in k^*$. We can now state our main result.

THEOREM. *Let J be an Albert division algebra over k arising form Tits' first construction. Let $x \in J$ be a Kummer element with $x^3 = \lambda$, for $\lambda \in k^*$. Then $J \simeq J(A, \lambda)$ for some central division algebra A of degree 3 over k.*

Before we can prove this, we need some results from Jordan theory. We list them here for convenience.

THEOREM 1 ([PR1, 2.7; PR2, 4.5]). *Let J be an Albert division algebra arising from Tits' first construction. Let L/k be a cubic cyclic extension contained in J. Then there exists a subalgebra D^+ of J, for a degree-3 central division algebra D over k, such that $L \hookrightarrow D^+$.*

THEOREM 2 ([J, Chap. IX, Theorem 22]). *Let J be an Albert algebra over k. If J contains a subalgebra D^+, for some central simple algebra D of degree 3 over k, then $J \simeq J(D, \mu)$, for some $\mu \in k^*$.*

THEOREM 3 ([PR3, Theorem 2′]). *Let E/k be a cyclic extension $\mathrm{Gal}(E/k) = \langle \sigma \rangle$. For $\gamma \in k^*$, denote by $E(\gamma)$ the cyclic algebra (E, σ, γ). Then for any nonzero $\lambda, \mu \in k$, the Albert algebras $J(E(\lambda), \mu)$ and $J(E(\mu)^o, \lambda)$ are isomorphic, where $E(\mu)^o$ is the opposite algebra of $E(\mu)$.*

Proof of Theorem. Let $x \in J$ be a Kummer element with $x^3 = \lambda$. Let $L = k(x)$ be the subalgebra generated by x. Since J is a division algebra and k contains third roots of unity, L is a cubic cyclic field extension of k. Applying Theorem 1 to this setup, we get a subalgebra D^+ of J which contains L. Then, by Theorem 2, $J \simeq J(D, \mu)$ for some $\mu \in k^*$. Since L is a cubic cyclic extension contained in D, we can write D as a cyclic algebra (L, γ) for some $\gamma \in k^*$. In the Brauer group of k we therefore have

$$[D] = [L] \cup [\gamma] = [\lambda] \cup [\gamma] = -[\gamma] \cup [\lambda].$$

Let E/k be the Kummer extension of k corresponding to γ and let B be the cyclic algebra (E, λ). Then, by the above computation in the Brauer group, we have

$$[D] = -[B] = [B^o],$$

where B^o denotes the opposite algebra of B. Thus $D \simeq B^o$ and $D^+ \simeq B^{o+}$. Therefore $J \simeq J(B^o, \mu)$. Now we appeal to the “flipping” theorem of Petersson and Racine, Theorem 3 above. We note that $B = E(\lambda)$ and $J \simeq J(E(\lambda)^o, \mu)$. Hence, by the above theorem, $J \simeq J(E(\mu), \lambda) = J(A, \mu)$, $A = E(\mu)$. This completes the proof.

Remark. Let J be as above. Given $S \subset J$, recall from [PR1] that the strong orthogonal complement of S is defined by

$$S^{\perp\!\perp} = \{x \in J\colon x \in S^{\perp}, x^{\#} \in S^{\perp}\},$$

$x^{\#} = x \times x$. Since $J \simeq J(E(\lambda)^o, \mu)$, the first Tits’ construction $A^+ = J(E, \mu)$ is a subalgebra of J and the Kummer element x in the theorem belongs to $A^{+^{\perp\!\perp}}$.

COROLLARY 1. *Let J be a first-Tits’-construction Albert division algebra and let $x \in J$ be a Kummer element with $x^3 = \lambda$. Then $g_3(J) = [D] \cup [\lambda]$ for some D.*

Proof. By the theorem, $J \simeq J(D, \lambda)$ for some D. The rest is just the definition of g_3.

COROLLARY 2. *Let $J_1 = J(A, \mu)$ and $J_2 = J(B, \nu)$ be Albert division algebras arising from Tits' first construction. If $g_3(J_1) = g_3(J_2)$ then $J_2 \simeq J(C, \mu)$ for some central simple algebra C over k of degree* 3.

Proof. Let L be a cubic extension that reduces (hence splits) J_1. Then

$$g_3(J_1 \otimes L) = 0 = g_3(J_2 \otimes L).$$

Thus L reduces J_2. But then, by [PR4, Corollary 3] and [PR2, (4.8)], it follows that J_2 contains an isomorphic copy of L. Therefore, J_1 and J_2 contain the same cubic subfields. Now, in J_1, $x = (0, 1, 0)$ is a Kummer element with $x^3 = \mu$. Set $L = k(x)$. Then, by the above, J_2 contains a copy of L and hence a Kummer element y with $y^3 = \mu$. Now the theorem applies and we have $J_2 \simeq J(C, \mu)$ for some C.

COROLLARY 3. *We have, for an Albert division algebra arising from Tits' first construction, the equality of sets*

$$\{\mu \in k^*: J \simeq J(A, \mu)\} = \{\mu \in k^*: x^3 = \mu, x \in J\}.$$

Proof. This is clear from the proof of Corollary 2 and the main result.

Remark. Note that by using the "flipping" theorem of Petersson and Racine (Theorem 3) we can push any of the symbols occurring in the decomposition of $g_3(J) = (a) \cup (b) \cup (c)$ into the last position. So the above corollary in fact shows that the symbols occurring in $g_3(J)$ correspond precisely to norms of the Kummer elements in J.

COROLLARY 4. *Let J be a Tits'-first-construction Albert division algebra over k and $g_3(J) = [A] \cup [\mu]$, for some central division algebra A of degree 3 over k and $\mu \in k^*$. Then $J \simeq J(B, \mu)$ for some degree-3 central division algebra B over k.*

Proof. Since J is a Tits' first construction, $J \simeq J(C, \nu)$ for appropriate parameters C and ν. Therefore we have, by the definition of the invariant g_3,

$$g_3(J) = g_3(J(C, \nu)) = g_3(J(A, \mu)).$$

Now the result follows by Corollary 2.

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